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DYNAMICS OF A FLUID IN A ROUGHNESS LAYER

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A study is made of the dynamics of a viscous Newtonian fluid in a roughness layer which takes into account the oscillation of the geometric structure of the layer space. The solution of modified Navier–Stokes equations is represented in the form of a three-scale expansion in powers of the geometric parameters of the roughness and describes the motion of the fluid throughout the layer (integral scale) and in the cells formed by roughness elements (local scales). The spatial averaging of the problem has been carried out; the system of equations which prescribes the integral dynamics of the fluid in the layer as a continuous medium with allowance for the contributions from the effects of local dynamics in the roughness cells and is a basis for construction of the models of turbulence in a strongly locally inhomogeneous medium has been given.

A great many works (for example, [1, 2]) are devoted to investigations of a turbulent boundary layer occurring in the case of plane-parallel flow of a fluid above a totally rough wall. The results obtained in them constitute the so-called laws of a logarithmic boundary layer ("wall laws")

$$\overline{u(z)} = \kappa^{-1} u_* \ln \frac{z}{z_0} , \quad b(z) = \frac{u_*^2}{\sqrt{c}} , \quad c = 0.046 , \quad \tau = \rho \overline{u'w'} , \quad z_0 \le z , \tag{1}$$

and are classical. The logarithmic laws (1) confirmed by numerous experiments have been obtained under the assumption of constancy of the turbulent momentum in the boundary layer and impermeability of the roughness level $z_0 \left(\tau(z) = \text{const}, \, k \frac{db}{dz} \Big|_{z=z_0} = 0 \right)$ to the turbulent motion. These assumptions do not distort very strongly the physical picture of the turbulent interaction of the fluid flow with a rough wall at heights that are at a large distance from the wall and in flows whose characteristic spatial scales are considerably larger than the level of roughness of the underlying surface. However the internal turbulent flow in the roughness layer and its influence on the characteristics of the external flow are excluded from consideration. The assumptions indicated are confirmed by experiments in tubes and channels and in the boundary layers above other rough surfaces when the heights at which the measurement in the boundary layer are carried out are an order of magnitude or more higher than the characteristic scales of inhomogeneity of the rough wall.

At the same time, a great many problems exist in which one must take into account the interaction of the fluid flow with the internal flow in the roughness layer and calculate their characteristics. These are, for example, the problems of the ability of vegetation and agrolandscapes to be blown by the wind and energy and mass transfer in these structures [3–6], the formation of waves and flows under the action of the wind above the free water surface [7], the efficiency of gas curtains in the case of cooling of porous surfaces [8], etc. In such problems, the upper boundary h of the roughness layer can no longer be considered to be impermeable to a turbulence flow and the turbulent-momentum flow cannot be considered to be constant in the external boundary layer. Accordingly, relations (1) also fail in the boundary layer above the roughness.

Furthermore, the Navier–Stokes equations themselves as the model of fluid dynamics in a roughness layer and the starting basis for construction of turbulence models hold only in inhomogeneity-free regions of space. A popular way out is in "virtual smearing" of the roughness in the layer and subsequent application of the methods of boundary-layer theory, statistical hydromechanics, or other techniques of investigation of turbulence without taking into account

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the local interactions in roughness cavities. Generally speaking, such an approach to solution of the problem of dynamics of a fluid in a high roughness and in the boundary layer above is incorrect.

Below we consider the dynamics of a viscous Newtonian fluid flow in a layer of a horizontally homogeneous roughness which is formalized based on the spatial averaging of Navier–Stokes equations. The results obtained provide a basis for construction of turbulence models in such a locally strongly inhomogeneous medium as a roughness. The subsequent application of the methods of statistical hydrodynamics [2, 5] to the averaged Navier–Stokes equations leads to a semiempirical model of turbulence in a roughness with an "instantaneous square law of resistance" and free boundaries and conditions on them, which describes turbulent flow in the roughness more completely. This distinguishes the model from the existing semiempirical model of turbulence in a roughness and in the boundary layer above it [3, 4] constructed on the basis of "virtual smearing" of roughness inhomogeneities in the layer space with fixed boundaries and boundary conditions, statistical averaging of Navier–Stokes equations, and account for the "averaged square law of resistance" of the roughness to the turbulent flow.

Roughness-Layer Model and the Hydrodynamic Problem of Fluid Flow. To construct a mathematical model of the interaction of a viscous-fluid flow with a roughness layer one must prescribe the region in which the hydrodynamic problem is determined and consider the external flow above the roughness in the layer $M_{hz} = \{z \ge h, x, y \in R^2\} \subset R^3$. A set free from roughness inhomogeneities for $z < h$ forms the region of internal flow. It is difficult to construct a general geometric model of this set because of the diversity of the shapes of inhomogeneities and of the laws of their distribution in space. Such distributions usually prescribe a random spatial structure to a roughness. To preserve the meaning of the average value of any quantity in the roughness layer in averaging the hydrodynamic problem one must construct geometric models of the distribution of inhomogeneities in space which possess ergodicity [9]. We restrict ourselves to a simple horizontally periodic roughness model which is ergodic in construction. Many roughnesses have a randomly periodic structure to which one extends, almost directly, all the results obtained for the hydrodynamic problem of turbulent flow in a horizontally periodic roughness. We also note that in the case of a roughness of arbitrary structure the latter can easily be approached by a finite set of randomly periodic structures.

We denote by $\Xi^{\varepsilon\eta\delta}$ the open horizontally periodic set which is free from inhomogeneities forming the roughness $\Gamma^{\varepsilon\eta\delta} = M_{0h} \setminus \Xi^{\varepsilon\eta\delta}$ in the layer $M_{0h} = \{x = \mathrm{col}\,(x_i, i = 1, 2, 3)\colon 0 < x_3 < h, x_{1,2} \in R^2\} \subset R^3$. We form a basic element of the periodic roughness $\Gamma^{\varepsilon\eta\delta}$, i.e., a single "tree" D. For simplicity we will assume that the tree D has n stages with the same number of "branches" of radius r "with leaves" in each stage and of interstage step d (r, $d << h$). It is clear that $n = [d^{-1}h]$; without loss of generality, we set $h = nd$. We identify the horizontal jth stage of the tree D with $\omega = (r \times r \times d)$, i.e., the cell at whose nodes we place the bodies π_{ij}, $i = 1, ..., 4$, bounded by the surfaces $\partial\pi_{ij}$. The filling of the layer M_{0h} with parts of the tree D which form the jth stage, $j = 1, ..., n$, will be expressed by the areas of the normal cross sections of these bodies: $s_{ij}^{\varepsilon} = s^{\varepsilon}(x_{ij}) < 1$ in a unit area of the side in the planes $(x_2 0 x_3)$ and $(x_1 0 x_3)$ along the roughness layer and $s_{ij}^{\eta} = s^{\eta}(x_{ij}) < 1$ in the plane $(x_1 0 x_2)$ across the roughness layer; x_{ij} are the vectors of the coordinates of the centers of the bodies π_{ij}. We set $s^{\varepsilon}(0) = s^{\eta}(0) = s^{\varepsilon}(h) = s^{\eta}(h) = 0$. The dynamic resistance of each stage to the flow will be expressed by the coefficients $c_{\mathrm{f}i}$, $i = 1, 2, 3$, in a unit of height h of the roughness layer. The tree D will be placed in the layer M_{0h} horizontally l-periodically ($r << 1$). Thus, the structure of the roughness $\Gamma^{\varepsilon\eta\delta}$ in the layer M_{0h} has the form of a horizontally periodic set of cells $\Omega = (l \times l \times h)$, inside each of which a vertical set of n cells $\omega = (r \times r \times d)$ is located. We will assume that the roughness degenerates in contraction of an ω cell to the point $(r \to 0,\ s_{ij} \to 0)$.

Let us introduce the dimensionless parameters $\varepsilon = rh^{-1}$, $\delta = lh^{-1}$, and $\eta = dh^{-1} = n^{-1}$ characterizing the relative dimensions of ω and Ω cells in the volume $(h \times h \times h)$ of the layer M_{0h}. To describe fluid flow in the free space of the layer, apart from the integral coordinates x, we introduce the local coordinates $y = \mathrm{col}\,(y_1, y_2)$, $y_i = \delta^{-1}x_i$, $i = 1, 2$, $z = \mathrm{col}\,(z_1, z_2, z_3)$, $z_i = \varepsilon^{-1}y_i$, $i = 1, 2$, and $z_3 = \eta^{-1}x_3$ in the Ω and ω cells respectively. In these coordinates, the equations of the oscillating dynamics of fluid flow in the region $\Xi^{\varepsilon\eta\delta}$ have the form

$$\partial_t u^{\varepsilon\eta\delta} + \langle u^{\varepsilon\eta\delta}, \nabla^{\varepsilon\eta\delta}\rangle\, u^{\varepsilon\eta\delta} = \nu\Delta^{\varepsilon\eta\delta} u^{\varepsilon\eta\delta} - \rho^{-1}\nabla^{\varepsilon\eta\delta} p^{\varepsilon\eta\delta}\,, \tag{2}$$

$$(\mathrm{div})^{\varepsilon\eta\delta}\, u^{\varepsilon\eta\delta} = 0\,, \tag{3}$$

$$u^{\varepsilon\eta\delta} = u\,(t,\, \delta^{-1}Px,\, \delta^{-1}\varepsilon^{-1}Px,\, \eta^{-1}Qx,\, x)\,, \quad x \in \Xi^{\varepsilon\eta\delta} \times [t_0, \infty)\,, \tag{4}$$

where $P = \begin{pmatrix} 100 \\ 010 \end{pmatrix}^*$ and $Q = (001)^*$ are the projectors of $R^3 \to R^2$ and R^1, and become classical Navier–Stokes equations in the external boundary layer $M_{hx_3} = \{x\colon h < x_3,\, x_{1,2} \in R^2\}$ adjacent to the roughness.

For system (2)–(4) we prescribe the following boundary and initial conditions:

(a) at the boundary of the inhomogeneities for $x \in \partial\Xi^{\varepsilon\eta\delta}$ and $t \in [t_0, \infty)$ we have the tensor relationship between the stresses in the fluid $\sigma^{\varepsilon\eta\delta}$ and the specific forces of the resistance $\tau_{\mathrm{w}}^{\varepsilon\eta\delta}$ of the inhomogeneities to the fluid motion expressed by the velocity of the flow at the inlet to the ω cell and the specific coefficient of resistance $c_{\mathrm{f}i}$

$$\frac{1}{\text{mes}\ \omega} \int\limits_{\partial\pi_{ij}} \Sigma^{\varepsilon\eta\delta}\, dz n_{\partial\pi} = \tau_{\mathrm{w}}^{\varepsilon\eta\delta} = \rho C_{\mathrm{f}} S_{ij}^{\pi} \left[u^{\varepsilon\eta\delta} \otimes u^{\varepsilon\eta\delta} \right], \tag{5}$$

$$\Sigma^{\varepsilon\eta\delta} = (\sigma_{ij})_{i,j=1,}^{3} = -\,(\delta_{ij}\, p^{\varepsilon\eta\delta} - 2\mu\delta_{ik}\delta_{jl} e_{kl}\,(u^{\varepsilon\eta\delta}))_{i,j,k,l=1}^{3}\,, \tag{6}$$

$$e^{\varepsilon\eta\delta} = e_{kl}\,(u^{\varepsilon\eta\delta}) = \frac{1}{2} \left(\frac{\partial u_k^{\varepsilon\eta\delta}}{\partial x_l} + \frac{\partial u_l^{\varepsilon\eta\delta}}{\partial x_k} \right)_{k,l=1}^{3},$$

$$C_{\mathrm{f}} = \text{diag}\,(c_{\mathrm{f}l}\,,\ l = 1, 2, 3)\,, \quad S_{ij}^{\pi} = \text{diag}(s_{ij}^{\varepsilon}, s_{ij}^{\varepsilon}, s_{ij}^{\eta})\,, \quad \left[u^{\varepsilon\eta\delta} \otimes u^{\varepsilon\eta\delta} \right] = \text{col}\,((u_i^{\varepsilon\eta\delta})^2)_{i=1}^{3}\,; \tag{7}$$

(b) in the hyperplane $M_0 = \{x\colon x_3 = 0,\, x_{1,2} \in R^2\}$ we have the condition of sticking of the flow to an impermeable wall

$$u^{\varepsilon\eta\delta} = 0\,, \quad p^{\varepsilon\eta\delta} = 0\,, \quad x \in M_0\,, \quad t \in [t_0, \infty)\,; \tag{8}$$

(c) in the hyperplane $M_h = \{x\colon x_3 = h,\, x_{1,2} \in R^2\}$ we have smooth joining of the characteristics of the internal and external flows

$$u^{\varepsilon\eta\delta}\,(Px,\, h - 0,\, t) = u\,(Px,\, h + 0,\, t)\,, \quad \partial u^{\varepsilon\eta\delta}\,(Px,\, h - 0,\, t) = \partial u\,(Px,\, h + 0,\, t)\,,$$

$$p^{\varepsilon\eta\delta}\,(Px,\, h - 0,\, t) = p\,(Px,\, h + 0,\, t)\,, \quad t \in [t_0, \infty)\,; \tag{9}$$

(d) the initial conditions are

$$u_0^{\varepsilon\eta\delta} = \tilde{u}_0^{\varepsilon\eta\delta} = \tilde{u}\,(t_0,\, \delta^{-1}Px,\, \delta^{-1}\varepsilon^{-1}Px,\, \eta^{-1}Qx,\, x)\,, \quad x \in \Xi^{\varepsilon\eta\delta}\,, \quad t_0 \in [t_0, \infty)\,, \tag{10}$$

$$p_0^{\varepsilon\eta\delta} = \tilde{p}_0^{\varepsilon\eta\delta} = \tilde{p}\,(t_0,\, \delta^{-1}Px,\, \delta^{-1}\varepsilon^{-1}Px,\, \eta^{-1}Qx,\, x)\,, \quad x \in \Xi^{\varepsilon\eta\delta}\,, \quad t_0 \in [t_0, \infty)\,. \tag{11}$$

The solution of problem (2)–(11) necessitates its spatial averaging [10, 11].

Averaged Hydrodynamic Problem. The averaged equations of the integral dynamics of a fluid in a roughness layer have the form

$$\partial_t u^{000} + \langle u^{000}, (I - C_s)\, \nabla_x \rangle\, u^{000} = \nu\, \langle \nabla_x, (I - C_s)\, \nabla_x \rangle\, u^{000} - \rho^{-1}\, (I - C_s)\, \nabla_x\, p^{000} + C_f S\, [u^{000} \otimes u^{000}]\ , \tag{12}$$

$$\langle (I - C_s)\, \nabla_x, u^{000} \rangle = 0\ , \tag{13}$$

$$u^{000}\, (x_1, x_2, h - 0, t) = u\, (x_1, x_2, h + 0, t)\ ,\quad \partial_x u^{000}\, (x_1, x_2, h - 0, t) = \partial_x u\, (x_1, x_2, h + 0, t)\ ,$$
$$p^{000}\, (x_1, x_2, h - 0, t) = p\, (x_1, x_2, h + 0, t)\ , \tag{14}$$

$$u^{000}\, (x_1, x_2, 0, t) = 0\ ,\quad p^{000}\, (x_1, x_2, 0, t) = 0\ , \tag{15}$$

$$x_1, x_2 \in (-\infty, \infty)\ ,\quad x_3 \in [0, h]\ ,\quad t \in [0, \infty)\ ,$$

where $C_s = \mathrm{diag}\left(\frac{s^\delta}{2}(1 + s^\varepsilon), \frac{s^\delta}{2}(1 + s^\varepsilon), s^\eta\right)$, $S = \mathrm{diag}\left(s^\delta, s^\delta, s^\eta\right)$, and $u(x_1, x_2, h + 0, t)$, $\partial_x u(x_1, x_2, h + 0, t)$, $p(x_1, x_2, h + 0, t)$ is the solution of the hydrodynamic problem above the roughness at the upper level of inhomogeneities.

Recovery of the stressed state of the averaged flow from the equations of its motion [(12 and (13)] leads to the rheological relation

$$\Sigma^{000} = -\, p^{000} \left(I - C_s + \frac{1}{p^{000}} \lim_{\vartheta x \to 0} \int_{\vartheta x} p^{000} \nabla_x C_s dx \right) + 2\mu\, (\dot{E}_0^{000} - C_s \nabla_x E_1^{000}) -$$
$$-\, \rho \lim_{\vartheta x \to 0} \frac{1}{\partial \vartheta x} \int_{\vartheta x} C_f S\, [E_1^{000} \otimes E_1^{000}]\, dx\ ,$$

$E_1^{000} = \mathrm{diag}\, (u_i^{000})_{i=1}^3$ which characterizes the averaged flow as a non-Newtonian anisotropic pseudocompressible fluid with a density and internal resistance dependent on the rate of compressive strain, the velocity of motion of the flow, and the characteristics of the geometric structure of the layer M_{0h}. In the limiting situation ($C_s \to 0$), as is easily seen, this rheological relation becomes a generalized Newton law for an isotropic liquid. A detailed investigation of the rheodynamics of the model medium constructed above is beyond the scope of the present paper. We only note that rheological relations of such form can also be obtained in averaging the motion of the fluid in porous media and suspensions [12, 13].

Let us briefly explain the procedure for constructing system (12)–(15). We transform system (2)–(11) to the problem with zero initial and boundary conditions

$$\partial_t u^{\varepsilon\eta\delta} - \nu\, \Delta_x^{\varepsilon\eta\delta} u^{\varepsilon\eta\delta} = F^{\varepsilon\eta\delta}\ , \tag{2a}$$

$$(\mathrm{div})^{\varepsilon\eta\delta} u^{\varepsilon\eta\delta} = 0\ , \tag{3a}$$

$$\left(\sigma_{ij}^{\varepsilon\eta\delta} + \delta_{ij}\, p^{\varepsilon\eta\delta} - 2\mu \delta_{ik} \delta_{jl} e_{kl}^{\varepsilon\eta\delta} \left(u^{\varepsilon\eta\delta}\right)\right)_{i,j,k,l=1}^{3} = 0\ ,\quad x \in \partial \Xi^{\varepsilon\eta\delta}\ ,\quad t \in [t_{0,}\, \infty)\ , \tag{4a}$$

$$u^{\varepsilon\eta\delta}\, (t, 0) = 0\ ,\quad u_0^{\varepsilon\eta\delta}\, (x) = 0\ , \tag{5a}$$

$$F^{\varepsilon\eta\delta} = -\langle u^{\varepsilon\eta\delta}, \nabla_x^{\varepsilon\eta\delta}\rangle\, u^{\varepsilon\eta\delta} - \rho^{-1}\,\nabla_x^{\varepsilon\eta\delta} p^{\varepsilon\eta\delta} + \partial_n \langle \delta_S (M_h), u\,(h+0)\rangle\, n_h +$$

$$+ \langle \delta_S (\partial \Xi^{\varepsilon\eta\delta}), \rho C_{\mathrm{f}} S^{\varepsilon\eta\delta} [u^{\varepsilon\eta\delta} \otimes u^{\varepsilon\eta\delta}]\rangle\, n_S + \delta\,(t - t_0)\, \tilde{u}_0^{\varepsilon\eta\delta} ,$$

$$x \in \Xi^{\varepsilon\eta\delta}, \quad t \in [t_{0,} \infty)\,, \quad t_0 \in [0, \infty)\,. \tag{6a}$$

Transformation of system (2)–(11) into the equivalent system (2a)–(6a) is carried out using the nonlinear integro-differential equation with a kernel by the Green operator function and the standardized vector function $F^{\varepsilon\eta\delta}$ of the Stokes linear problem with homogeneous initial-boundary conditions [14].

To average problem (2a)–(6a) we apply the method of asymptotic expansions [10, 11]. Assuming that the integral x and local y and z variables are independent, we represent the differential operator of the gradient in the form

$$\nabla^{\varepsilon\eta\delta} = \nabla_x + \delta^{-1} P^* \nabla_y + \delta^{-1}\varepsilon^{-1} P^* \nabla_z + \eta^{-1} Q^* \nabla_z\,. \tag{16}$$

The expressions for the remaining differential operators appearing in system (2)–(11) can easily be obtained from (16) and the representations $\Delta\,(*) = \langle \nabla, \nabla\rangle(*)$ and $\operatorname{div}\,(*) = \langle \nabla, (*)\rangle$.

The solution (4) of system (2)–(11) is sought in the form of a series in powers of the parameters δ, η, and ε:

$$u^{\varepsilon\eta\delta}\,(x, y, z, t) = u^{000}\,(x, t) + \delta\,(u^{001}\,(Qx, y, t) + \varepsilon\,(u^{101}\,(y, Pz, t) + \ldots) + \ldots) +$$

$$+ \eta\,(u^{010}\,(y, Qz, t) + \ldots) + \ldots\,. \tag{17}$$

Substituting (16) and (17) and the corresponding expressions for the remaining operators into system (2a)–(6a) and equating subsequently the coefficients of the parameters δ, η, and ε of the same power on both sides of the expansions, we obtain the system of interrelated equations of the fluid dynamics and the initial-boundary conditions in the layer $[0, h]$ (integral scale) and in the ω and Ω cells (local scales). In subsequent constructions, we restrict ourselves to the employment of the terms of expansion which correspond just to the zero and first powers of each parameter. Thus, the system corresponding to the zero powers of the parameters δ, η, and ε is represented by the equations

$$\partial_t u^{000} + \langle u^{000}, \nabla_x\rangle\, u^{000} + \langle P u^{000}, P^* \nabla_y\rangle\, u^{001} + \langle P u^{000}, P^* \nabla_y\rangle\, u^{101} +$$

$$+ \langle Q u^{000}, Q^* \nabla_{Qz}\rangle\, u^{010} = \nu\,(\Delta_x u^{000} + (\langle \nabla_x, P^* \Delta_y\rangle + \langle P^* \Delta_y, \nabla_x\rangle)\, u^{001} +$$

$$+ (\langle \nabla_x, P^* \nabla_{Pz}\rangle + \langle P^* \nabla_{Pz}, \nabla_x\rangle)\, u^{101} + (\langle \nabla_x, Q^* \nabla_{Qz}\rangle + \langle Q^* \nabla_{Qz}, \nabla_x\rangle)\, u^{010}) -$$

$$- \rho^{-1}\,(\nabla_x p^{000} + P^* \nabla_y p^{001} + P^* \nabla_{Pz} p^{101} + Q^* \nabla_{Qz} p^{010}) + C_{\mathrm{f}} S\,[u^{000} \otimes u^{000}]\,, \tag{18}$$

$$\langle \nabla_x, u^{000}\rangle + \langle P^* \nabla_y, P u^{001}\rangle + \langle P^* \nabla_{Pz}, P u^{101}\rangle + \langle Q^* \nabla_{Qz}, Q u^{010}\rangle = 0\,. \tag{19}$$

Equations corresponding to the first powers of the parameters δ, η, and ε and the corresponding expressions for the boundary and initial conditions have analogous representations.

The resultant system is open since it contains a larger number of the sought components of expansion of the solution of the hydrodynamic problem than the number of equations and of the conditions corresponding to them. Therefore, the problem of closing arises for this system; it implies that smaller-scale components of the solution of the hydrodynamic problem are represented by larger-scale components and is analogous to the problem of closing in the statistical mechanics of turbulence [2].

Averaging and the Influence of the Geometry of the Roughness Layer. Let us apply operations of averaging over the sides of the ω and Ω cells to Eqs. (18) and (19) and the homogeneous boundary and initial conditions corresponding to these equations:

$$(*)^{\wedge} = \frac{1}{\text{mes } P\omega} \int_{P\omega} (*)\, dz\,, \quad (*)^{\vee} = \frac{1}{\text{mes } P\Omega} \int_{P\Omega} (*)\, dy\,, \quad (*)^{\cap} = \frac{1}{\text{mes } Q\omega} \int_{Q\omega} (*)\, dQz\,,$$

$$\text{mes } P\omega = r^2\,, \quad \text{mes } Q\omega = d\,, \quad \text{mes } P\Omega = l^2\,.$$

It is easily seen that $(u^{000})^{\wedge} = (u^{000})^{\vee} = (u^{000})^{\cap} = u^{000}$ and, passing from integration over the area of the cell to integration along its contour, we obtain

$$\left(\frac{\partial u_i^{101}}{\partial z_j}\right)^{\wedge,\vee,\cap} = \delta_{ij}\left(l^{-2} r^{-2} d^{-1} \int_{Qz\cup P\Omega\cup P\omega} \left(\langle n_j^{+}, u_i^{101}\, ({}^{+}S_{\omega}^{j})\rangle - \langle n_j^{-}, u_i^{101}\, ({}^{-}S_{\omega}^{j})\rangle\right) dz dy = \right.$$

$$= l^{-2} r^{-1} \int_{P\Omega} \left(\Pr\left(u_i^{101}\, ({}^{+}S_{\omega}^{i})\right)^{\wedge} - \Pr\left(u_i^{101}\, ({}^{-}S_{\omega}^{i})\right)^{\wedge}\right) dy \cong - l^{-2} \int_{P\Omega} \alpha_{ii}^{\varepsilon} \frac{\partial u_i^{001}}{\partial y_i}\, dy =$$

$$= - \alpha_{ii}^{\varepsilon}\, l^{-1} \left(\Pr\left(u_i^{001}\, ({}^{+}S_{\Omega}^{i})\right)^{\vee} - \Pr\left(u_i^{001}\, ({}^{-}S_{\Omega}^{i})\right)^{\vee}\right) \cong \begin{cases} - \alpha_{ii}^{\varepsilon}\, \alpha_{ii}^{\delta} \dfrac{\partial u_i^{000}}{\partial x_i}\,, & i = j\,, \\ 0\,, & i \neq j\,, \end{cases}$$

and hence the horizontal averaging of local motions of the fluid in the ω cells leads to the occurrence of additional horizontal gradients of the averaged horizontal velocities and to additions to the convective acceleration of a fluid particle in the averaged flow. By virtue of the *l*-periodicity of the roughness structure on the free space of the Ω cell we have the restoration of the loss of the horizontal components of the velocity vector which occurs in the case of flow through the ω cell.

The averaging of local motions in the ω cell across the layer also leads to the appearance of the additional component of the convective acceleration of the particle in the averaged flow

$$\left(\frac{\partial u_i^{010}}{\partial z_3}\right)^{\wedge,\vee,\cap} = d^{-1}\left(\langle n_3^{+}, (u_i^{010}\, ({}^{+}S_{\omega}^{i}))^{\cap}\rangle - \langle n_3^{-}, (u_i^{010}\, ({}^{-}S_{\omega}^{i}))^{\cap}\rangle\right) \cong \begin{cases} 0\,, & i \neq 3\,, \\ - \alpha_{i3}^{\eta} \dfrac{\partial u^{000}}{\partial x_3}\,, & i = 3\,, \end{cases}$$

and to a defect of the vertical component of the velocity vector in the case of flow through the ω cell. Here $\langle n_j^{+(-)}, u_i^{101(001)}({}^{+(-)}S_{\omega(\Omega)}^{j})\rangle$ is the projection of the *i*th component of the vector of the flow velocity to the sides ${}^{+(-)}S_{\omega(\Omega)}^{j}$ of the ω(Ω) cell which are opposite to *j* and $n_j^{+(-)}$ are the normals to these sides.

We have the same situation (up to a sign) in averaging for both the horizontal pressure gradient in the ω cell and the vertical pressure gradient in the layer [0, *h*]:

$$\left(P\nabla_z\, p^{101}\right)^{\wedge,\vee,\cap} = l^{-2} r^{-2} d^{-1} \int_{Qz\cup P\omega\cup P\Omega} \left(n_j^{+}\, (p^{101})^{+} - n_j^{-}\, (p^{101})^{-}\right) dz dy = \beta_j^{\delta} \frac{\partial p^{000}}{\partial x_j}\,,$$

$$\left(Q\nabla_z p^{010}\right)^{\wedge,\vee,\cap} \cong \beta_3^{\eta}\,\frac{\partial p^{000}}{\partial x_3}\,.$$

The averaging of the medium-scale components of the velocity $P\nabla_y u^{001}$ and the pressure gradient $P\nabla_y p^{001}$ over the Ω cells also leads to a change in the convective acceleration, the potential component of the velocity, and the pressure in the averaged flow, since

$$\left(\frac{\partial u_i^{001}}{\partial y_j}\right)^{\vee,\cap} = \begin{cases} 0\,, & i \neq j\,, \\ \alpha_{ii}^{\delta}\,\dfrac{\partial u_i^{000}}{\partial x_i}\,, & i = j\,; \end{cases} \qquad \left(\frac{\partial p^{001}}{\partial y_3}\right)^{\vee,\cap} = \begin{cases} 0\,, & i \neq 3\,, \\ \beta_3^{\delta}\,\dfrac{\partial p^{000}}{\partial x_3}\,, & i = 3\,. \end{cases}$$

The coefficients $\alpha_{ii}^{\varepsilon}$, α_{ii}^{δ}, β_i^{ε}, β_i^{δ}, $i = 1,\ 2$, α_{33}^{η}, and β_3^{η} represent functions which depend on the geometric characteristics of the inhomogeneities, the parameters of their location in the layer, and the difference of the flow velocities and the pressure in the corresponding cells. In accordance with the conditions at the boundaries of the roughness layer, it is natural to take

$$\alpha_{ii}^{\varepsilon(\delta)}(0) = \alpha_{ii}^{\varepsilon(\delta)}(h) = 0\,,\quad \beta_i^{\varepsilon(\delta)}(0) = \beta_i^{\varepsilon(\delta)}(h) = 0\,,\quad \alpha_{33}^{\eta}(0) = \alpha_{33}^{\eta}(h) = 0\,,\quad \beta_{33}^{\eta}(0) = \beta_{33}^{\eta}(h) = 0\,.$$

Both solutions of the problems in local cells and experimental investigations of flow in the roughness are required to establish the form of three functions. From dimensional considerations we take

$$\alpha_{ii}^{\varepsilon} = \beta_i^{\varepsilon} = s^{\varepsilon}\,,\quad \alpha_{33}^{\eta} = \beta_3^{\eta} = s^{\eta}\,,\quad \alpha_{ii}^{\delta} = \beta_i^{\delta} = \frac{s^{\delta}}{2}\,,$$

where

$$\frac{1}{\text{mes}\,Q\omega\,\sqrt{\text{mes}\,P\omega}} \int\limits_{\text{mes}Q\omega\sqrt{\text{mes}P\omega}} \sum_{i,j=1}^{2} s^{\eta}(x_1, x_{2i}, x_{3j}, z, y)\, dQz = s^{\eta}(y, Qx)\,,$$

$$\frac{1}{\text{mes}\,P\omega} \int\limits_{P\omega} \sum_{i,j=1}^{2} s^{\varepsilon}(x_{1i}, x_{2j}, x_3, z, y)\, dPz = s^{\varepsilon}(y, Qx)\,,\quad \frac{1}{\text{mes}\,P\Omega} \int\limits_{P\Omega} s^{\varepsilon}(x, y)\, dy = s^{\delta}(x)\,.$$

Thus, by virtue of the inhomogeneity of the geometric space structure of the layer $[0, h]$ as a result of the averagings we have

$$\left[\langle u^{000}, \nabla_x\rangle\, u^{000} + \langle Pu^{000}, P^*\nabla_y\rangle\, u^{001} + \langle Pu^{000}, P^*\nabla_z\rangle\, Pu^{101} + \langle Qu^{000}, Q^*\nabla_z\rangle\, u^{010}\right]^{\wedge,\vee,\cap} =$$

$$= \langle u^{000}, (I - C_s)\,\nabla_x\rangle\, u^{000}\,,$$

$$\Big[\Delta_x u^{000} + \left(\langle\nabla_x, P^*\nabla_y\rangle + \langle P^*\nabla_y, \nabla_x\rangle\right)u^{001} + \left(\langle\nabla_x, P^*\nabla_{Pz}\rangle + \langle P^*\nabla_{Pz}, \nabla_x\rangle\right)u^{101} +$$

$$+ \left(\langle\nabla_x, Q^*\nabla_{Qz}\rangle + \langle Q^*\nabla_{Qz}, \nabla_x\rangle\right)u^{010}\Big]^{\wedge,\vee,\cap} = \langle\nabla_x, (I - C_s)\,\nabla_x\rangle\, u^{000}\,,$$

$$\left[\nabla_x p^{000} + \left(P^*\nabla_y p^{001}\right) + \left(P^*\nabla_z p^{101}\right) + Q^*\nabla_z p^{010}\right]^{\wedge,\vee,\cap} = (I - C_s)\,\nabla_x p^{000}\,,$$

$$\left(\langle \delta_S (\partial \Xi^{\varepsilon\eta\delta}), \tau_w^{\varepsilon\eta\delta}\rangle\, n_S\right)^{\wedge,\vee,\cap} = \rho C_f S\left[u^{000} \otimes u^{000}\right],$$

$$\left[\langle \nabla_x, u^{000}\rangle + \langle P^* \nabla_y, Pu^{001}\rangle + \langle P^* \nabla_{Pz}, Pu^{101}\rangle + \langle Q^* \nabla_{Qz}, Qu^{010}\rangle\right]^{\wedge,\vee,\cap} = \langle (I - C_s)\, \nabla_x, u^{000}\rangle$$

and without dwelling on the procedure of averaging of the remaining linear components in (18)–(19) and of the remaining initial-boundary conditions, we arrive at system (12)–(15).

The averaged equations of the local dynamics of the flow in the Ω and ω cells are constructed analogously. These equations, together with the averaged Navier–Stokes equations for the entire roughness layer, form a complete system of Navier–Stokes equations which formalizes the hydrodynamics of a viscous fluid in the layer of a periodic roughness in traditional terms. Applying the methods of the statistical hydromechanics of turbulence to system (12)–(15) [2] and taking into account that, because of the periodicity of the geometric structure of the roughness, all the horizontal additions to the convective acceleration, the potential velocity component, and the pressure in the averaged flow vanish for stationary fluid motions, including periodic ones, one can construct the models of turbulence in the layer of a periodic roughness. Thus, the model of turbulence in homogeneous vegetation with free conditions at the boundaries and a free position of the lower boundary of the turbulent zone (determinable by solution of the problem) that realizes the instantaneous square law of resistance of the vegetation to the motion of an air flow has been constructed and investigated in [5, 6].

NOTATION

$\overline{u(z)}$, average horizontal component of the vector of the flow velocity; $b(z)$, intensity of turbulence in the boundary layer; τ, turbulent (Reynolds) tangential stress in the flow; u' and w', pulsation horizontal and vertical components of the vector of the flow velocity; $\overline{u'w'}$, correlation of the horizontal and vertical pulsations of the velocity vector; u_*, dynamic velocity in the boundary layer (characteristic velocity scale); t, time; t_0, initial instant of time; μ, ν, and ρ, coefficients of dynamic and kinematic viscosity of the fluid and its density; $\kappa = 0.4$, Kármán constant; z_0, level of roughness of the underlying surface (mean statistical height of the inhomogeneities forming the roughness of the wall in flow); z, vertical coordinate; k, coefficient of turbulent viscosity; h, mean statistical height of the roughness inhomogeneities; $u^{\varepsilon\eta\delta}$ and $p^{\varepsilon\eta\delta}$, velocity of the fluid flow and deviation of the total pressure from an equilibrium p_{st} in roughness cavities; R^m, $m = 1, 2, 3$, m-dimensional Euclidean space; s^η, average surface of inhomogeneities in a unit square of the vertical side of the ω cell in the direction across the layer; s^ε, average surface of inhomogeneities of a unit square of the horizontal side of the ω cell in the direction along the layer; s^δ, average surface of inhomogeneities in a unit square of the horizontal side of the Ω cell in the direction along the layer; Pr (∗), projection of the vector (∗) on the sides of the cell; C_s and S, tensors of the characteristics of the averaged relative blocking of the layer M_{0h} by the roughness; $(I - C_s)$, tensor of the characteristics of the relative "transparency" of the layer M_{0h}; $\Sigma^{\varepsilon\eta\delta}$ and Σ^{000}, stress tensors in the nonaveraged and averaged flows; E_0^{000}, standard tensor of the strain rate of the averaged flow as a Newtonian isotropic medium; $\tau_w^{\varepsilon\eta\delta}$, specific force of resistance of the inhomogeneities in roughness cells to the flow; C_f, tensor of the coefficients of dynamic resistance of the roughness; mes A, measure of the set A; I, operator unit; $\langle *, \&\rangle$ and $[* \otimes \&] = \mathrm{col}\,((*)_i(\&)_i)_{i=1}^3$, inner and outer Kronecker products of the vectors (∗) and (&); $\vartheta(x)$ and $\partial\vartheta(x)$, vicinity of the point x and its boundary; $\delta(*)$, Dirac delta function of the variable (∗); $\langle \delta_s(x), (**)\rangle = \int_D \langle \delta_s(x), (**(x))\rangle\, dx = \int_S (**(x))\, dS$, $\partial_n \langle \delta_s(x), (**)\rangle(***) = \int_D (***(x))\, \partial_n\langle \delta_s(x), (**(x))\rangle\, dx = \int_S \partial_n(***(x))(**(x))dS$, generalized functions of the single and double layers for the surface $\partial D = S$, i.e., the boundary of the corresponding region D;

n_S, normal to *S*; (∗∗) and (∗∗∗), vector functions prescribed at *S* and *D*. Subscripts: st, equilibrium; s, area; w, wall; f, fluid.

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